Modeling Loan Portfolios

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March, 2024

In this white paper we will model a loan portfolio in continuous-time. To that end we will work through the following hypothetical problem...

Our Hypothetical Problem

ABC Bank originates loans and then carries those loans on its balance sheet. We are given the following go-forward model assumptions...

Table 1: Model Assumptions (Dollars in thousands)

Description	Value
Loan portfolio at time zero ($)	1,500
Monthly loan originations at time zero ($)	80
Weighted-average loan term in years (#)	5.00
Weighted-average loan life in years (#)	3.00
Number of annual periods (#)	12
Loan originations annualized growth rate (%)	4.50
After-tax return on assets (%)	2.00

Our task is to answer the following questions...

Question 1: What is loan portfolio principal balance at the end of year 2?

Question 2: What is loan portfolio principal balance at the end of year 3?

Question 3: Reconcile the change in loan portfolio principal balance above.

Question 4: What is net income in year 3?

Individual Loan Mathematics

We will define the variables s, t, m and n as time in years. The relationship between these time variables are...

$$s \le t \text{ ...and... } s \le m \le n \tag{1}$$

We will define the variable L_t^s to be loan principal balance at time t on a loan originated at time $s \le t$, and the variable ω to be the loan's weighted average life in years. The equation for loan principal balance is... [1]

$$L_t^s = L_s^s \,\mathrm{Exp}\left\{ - \lambda \,(t-s) \right\} \text{ ...where... } \lambda = \frac{1}{\omega} \tag{2}$$

If loan originations grow at the continuous-time rate μ then the equation for loan originations at time s as a function of loan originations at time zero is...

$$L_s^s = L_0^0 \,\mathrm{Exp}\left\{ \mu \, s \right\} \text{ ...where... } \mu = \ln \left(1 + \text{Annualized growth rate} \right) \tag{3}$$

Using Appendix Equation (33) below, we can rewrite Equation (2) above as...

$$L_t^s = L_0^0 \,\mathrm{Exp}\left\{ (\mu + \lambda)\, s \right\} \mathrm{Exp}\left\{ - \lambda \, t \right\} \tag{4}$$

The derivative of Equation (4) above with respect to time is...

$$\delta L_t^s = -\lambda\, L_0^0 \,\mathrm{Exp}\left\{(\mu+\lambda)\, s\right\} \mathrm{Exp}\left\{-\lambda\, t\right\} \delta t \tag{5}$$

We will define the variable $X_{m,n}^s$ to be loan principal repayments over the time interval $[m,n]$ on a loan originated at time s. Using Equation (5) above, the equation for loan principal repayments is...

$$X_{m,n}^s = \int_m^n \lambda\, L_0^0 \,\mathrm{Exp}\left\{(\mu+\lambda)\, s\right\} \mathrm{Exp}\left\{-\lambda\, v\right\} \delta v \tag{6}$$

Using Appendix Equation (34) below, the solution to Equation (6) above is...

$$X_{m,n}^s = L_0^0 \,\mathrm{Exp}\left\{(\mu+\lambda)\, s\right\} \left(\mathrm{Exp}\left\{-\lambda\, m\right\} - \mathrm{Exp}\left\{-\lambda\, n\right\}\right) \tag{7}$$

We will define the variable $N_{m,n}^s$ to be net income recognized over the time interval $[m,n]$ on a loan originated at time s and the variable θ to be the after-tax return on assets. Using Equation (5) above, the equation for after-tax net income is...

$$N_{m,n}^s = \int_m^n \theta L_0^0 \,\mathrm{Exp}\left\{\mu\, s\right\} \mathrm{Exp}\left\{-\lambda\,(v-m)\right\} \delta v \tag{8}$$

Using Appendix Equation (34) below, the solution to Equation (8) above is...

$$N_{m,n}^s = \theta\, L_0^0 \,\mathrm{Exp}\left\{(\mu+\lambda)\, s\right\} \left(\mathrm{Exp}\left\{-\lambda\, m\right\} - \mathrm{Exp}\left\{-\lambda\, n\right\}\right) \lambda^{-1} \tag{9}$$

Loan Portfolio Mathematics

Since we are moving from discrete-time to continuous-time we need an equation for annualized loan originations at time zero. Given that periodic loan originations at time zero is L_0^0, the equation for annualized loan originations at time zero is...

$$\text{Annualized loan originations} = L_0^0 \times \Delta^{-1} \;\; \text{...where...} \;\; \Delta = \text{Period length in years} \tag{10}$$

We will define the variable L_t to be loan portfolio balance at time t. Using Equations (4) and (10) above, the equation for loan portfolio balance at time t is...

$$L_t = L_0 \,\mathrm{Exp}\left\{-\lambda\, t\right\} + \int_0^t \Delta^{-1} L_0^0 \,\mathrm{Exp}\left\{(\mu+\lambda)\, v\right\} \mathrm{Exp}\left\{-\lambda\, t\right\} \delta v \tag{11}$$

Using Appendix Equation (35) below, the solution to Equation (11) above is...

$$L_t = L_0 \,\mathrm{Exp}\left\{-\lambda\, t\right\} + \Delta^{-1} L_0^0 \left(\mathrm{Exp}\left\{\mu\, t\right\} - \mathrm{Exp}\left\{-\lambda\, t\right\}\right) \left(\mu+\lambda\right)^{-1} \tag{12}$$

We will define the variable $Y_{m,n}$ to be cumulative loan originations over the time interval $[m,n]$. Using Equation (10) above, the equation for cumulative loan originations is...

$$Y_{m,n} = \int_m^n \Delta^{-1} L_0^0 \,\mathrm{Exp}\left\{\mu\, v\right\} \delta v \tag{13}$$

Using Appendix Equation (36) below, the solution to Equation (13) above is...

$$Y_{m,n} = \Delta^{-1} L_0^0 \left(\mathrm{Exp}\left\{\mu\, n\right\} - \mathrm{Exp}\left\{\mu\, m\right\}\right) \mu^{-1} \tag{14}$$

We will define the variable $X_{m,n}$ to be loan portfolio principal repayments over the time interval $[m, n]$. Using Equation (12) above, the equation for loan portfolio principal repayments is...

$$X_{m,n} = \int_m^n \lambda \left[L_0 \,\mathrm{Exp}\left\{ -\lambda\, v \right\} + \Delta^{-1} L_0^0 \left(\mathrm{Exp}\left\{ \mu\, v \right\} - \mathrm{Exp}\left\{ -\lambda\, v \right\} \right) \left(\mu + \lambda \right)^{-1} \right] \delta v \tag{15}$$

We will make the following integral definitions...

$$I(1) = \int_m^n L_0 \,\mathrm{Exp}\left\{ -\lambda\, v \right\} \delta v \quad \text{...and...} \quad I(2) = \int_m^n \Delta^{-1} L_0^0 \left(\mathrm{Exp}\left\{ \mu\, v \right\} - \mathrm{Exp}\left\{ -\lambda\, v \right\} \right) \left(\mu + \lambda \right)^{-1} \delta v \tag{16}$$

Using the integral definitions in Equation (16) above, we can rewrite Equation (15) above as...

$$X_{m,n} = \lambda \left(I(1) + I(2) \right) \tag{17}$$

Using Appendix Equation (37) below, the solution to the first integral in Equation (17) above is...

$$I(1) = L_0 \left(\mathrm{Exp}\left\{ -\lambda\, m \right\} - \mathrm{Exp}\left\{ -\lambda\, n \right\} \right) \lambda^{-1} \tag{18}$$

Using Appendix Equation (38) below, the solution to the second integral in Equation (17) above is...

$$I(2) = \Delta^{-1} L_0^0 \left(\mu + \lambda \right)^{-1} \left[\left(\mathrm{Exp}\left\{ \mu\, n \right\} - \mathrm{Exp}\left\{ \mu\, m \right\} \right) \mu^{-1} + \left(\mathrm{Exp}\left\{ -\lambda\, n \right\} \right) - \mathrm{Exp}\left\{ -\lambda\, m \right\} \right) \lambda^{-1} \right] \tag{19}$$

We will define the variable $N_{m,n}$ to be loan portfolio net income recognized over the time interval $[m, n]$. Using Equation (15) above as our guide, the equation for after-tax net income is...

$$N_{m,n} = \int_m^n \theta \left[L_0 \,\mathrm{Exp}\left\{ -\lambda\, v \right\} + \Delta^{-1} L_0^0 \left(\mathrm{Exp}\left\{ \mu\, v \right\} - \mathrm{Exp}\left\{ -\lambda\, v \right\} \right) \left(\mu + \lambda \right)^{-1} \right] \delta v \tag{20}$$

Using the solution to Equation (15) above as our guide, the solution to Equation (20) above is...

$$N_{m,n} = \theta \left(I(1) + I(2) \right) \tag{21}$$

The Answers To Our Hypothetical Problem

Using Equation (10) above and the model parameters in Table 1 above, the equation for model parameter Δ is...

$$\Delta = \frac{1}{12} = 0.0833 \tag{22}$$

Using Equation (2) above and the model parameters in Table 1 above, the equation for model parameter λ is...

$$\lambda = \frac{1}{3.00} = 0.3333 \tag{23}$$

Using Equations (12) and the model parameters in Table 1 above, the equation for model parameter μ is...

$$\mu = \ln \left(1 + 0.0450 \right) = 0.0440 \tag{24}$$

Using Equation (18) above, the value of integral one over the time interval $[2, 3]$ is...

$$I(1) = 1,500 \times \left(\mathrm{Exp}\left\{ -0.3333 \times 2.00 \right\} - \mathrm{Exp}\left\{ -0.3333 \times 3.00 \right\} \right) \times 0.3333^{-1} = 654.92 \tag{25}$$

Using Equation (19) above, the value of integral two over the time interval $[2, 3]$ is...

$$I(2) = 0.0833^{-1} \times 80 \times \left(0.0440 + 0.3333\right)^{-1} \times \left[\left(\text{Exp}\left\{0.0440 \times 3.00\right\} - \text{Exp}\left\{0.0440 \times 2.00\right\}\right) \times 0.0440^{-1} + \left(\text{Exp}\left\{-0.3333 \times 3.00\right\}\right) - \text{Exp}\left\{-0.3333 \times 2.00\right\}\right) \times 0.3333^{-1}\right] = 1,729.45 \tag{26}$$

Question 1: What is loan portfolio principal balance at the end of year 2?

Using Equations (12), (22), (23) and (24) above and the model parameters in Table 1 above, the answer to the question is...

$$L_2 = 1,500 \times \text{Exp}\left\{-0.3333 \times 2.00\right\} + 0.0833^{-1} \times 80 \times \left(\text{Exp}\left\{0.0440 \times 2.00\right\} - \text{Exp}\left\{-0.3333 \times 2.00\right\}\right)\left(0.0440 + 0.3333\right)^{-1} = 2,242.14 \tag{27}$$

Question 2: What is loan portfolio principal balance at the end of year 3?

$$L_3 = 1,500 \times \text{Exp}\left\{-0.3333 \times 3.00\right\} + 0.0833^{-1} \times 80 \times \left(\text{Exp}\left\{0.0440 \times 3.00\right\} - \text{Exp}\left\{-0.3333 \times 3.00\right\}\right)\left(0.0440 + 0.3333\right)^{-1} = 2,519.10 \tag{28}$$

Question 3: Reconcile the change in loan portfolio principal balance above.

Using Equation (14) above, loan originations in year 3 are...

$$Y_{2,3} = 0.0833^{-1} \times 80 \times \left(\text{Exp}\left\{0.0440 \times 3.00\right\} - \text{Exp}\left\{0.0440 \times 2.00\right\}\right) \times 0.0440^{-1} = 1,071.76 \tag{29}$$

Using Equation (17) above and the intergral solutions in Equations (25) and (26) above, loan portfolio principal repayments in year 3 are...

$$X_{2,3} = 0.3333 \times \left(654.92 + 1,729.45\right) = 794.79 \tag{30}$$

Using Equations (27), (28), (29) and (30) above, the answer to the question is...

Description	Amount
Beginning balance	2,242.14
Loan originations	1,071.76
Principal repayments	-794.79
Ending balance	2,519.10

Question 4: What is net income in year 3?

Using Equation (21) above and the intergral solutions in Equations (25) and (26) above, loan portfolio principal repayments in year 3 are...

$$N_{2,3} = 0.0200 \times \left(654.92 + 1,729.45\right) = 47.69 \tag{31}$$

Appendix

A. The solution to the following integral is...

$$\int_a^b \text{Exp}\left\{c\,t\right\}\delta t = \frac{1}{c}\left(\text{Exp}\left\{b\,t\right\} - \text{Exp}\left\{a\,t\right\}\right) \tag{32}$$

B. Using Equations (2) and (3) above, the solution to the following equation is...

$$L_t^s = L_0^0 \,\text{Exp}\left\{\mu\, s\right\} \text{Exp}\left\{-\lambda\,(t-s)\right\} = L_0^0 \,\text{Exp}\left\{(\mu+\lambda)\, s\right\} \text{Exp}\left\{-\lambda\, t\right\} \tag{33}$$

C. Using Equation (32) above, the solution to the following integral is...

$$\begin{aligned}
I &= \int_m^n L_0^0 \,\text{Exp}\left\{(\mu+\lambda)\, s\right\} \text{Exp}\left\{-\lambda\, v\right\} \delta v \\
&= L_0^0 \,\text{Exp}\left\{(\mu+\lambda)\, s\right\} \int_m^n \text{Exp}\left\{-\lambda\, v\right\} \delta v \\
&= -\frac{1}{\lambda}\, L_0^0 \,\text{Exp}\left\{(\mu+\lambda)\, s\right\} \left(\text{Exp}\left\{-\lambda\, n\right\} - \text{Exp}\left\{-\lambda\, m\right\}\right) \\
&= \frac{1}{\lambda}\, L_0^0 \,\text{Exp}\left\{(\mu+\lambda)\, s\right\} \left(\text{Exp}\left\{-\lambda\, m\right\} - \text{Exp}\left\{-\lambda\, n\right\}\right)
\end{aligned} \tag{34}$$

D. We want to find the solution to the following integral...

$$\begin{aligned}
I &= \int_0^t \Delta^{-1} L_0^0 \,\text{Exp}\left\{(\mu+\lambda)\, v\right\} \text{Exp}\left\{-\lambda\, t\right\} \delta v \\
&= \Delta^{-1} L_0^0 \,\text{Exp}\left\{-\lambda\, t\right\} \int_0^t \text{Exp}\left\{(\mu+\lambda)\, v\right\} \delta v \\
&= \Delta^{-1} L_0^0 \,\text{Exp}\left\{-\lambda\, t\right\} \frac{1}{\mu+\lambda} \left(\text{Exp}\left\{(\mu+\lambda)\, t\right\} - \text{Exp}\left\{(\mu+\lambda)\, 0\right\}\right) \\
&= \Delta^{-1} L_0^0 \,\text{Exp}\left\{-\lambda\, t\right\} \left(\text{Exp}\left\{(\mu+\lambda)\, t\right\} - 1\right) \left(\mu+\lambda\right)^{-1}
\end{aligned} \tag{35}$$

E. We want to find the solution to the following integral...

$$\begin{aligned}
I &= \int_m^n \Delta^{-1} L_0^0 \,\text{Exp}\left\{\mu\, v\right\} \delta v \\
&= \Delta^{-1} L_0^0 \int_m^n \text{Exp}\left\{\mu\, v\right\} \delta v \\
&= \Delta^{-1} L_0^0 \left(\text{Exp}\left\{\mu\, n\right\} - \text{Exp}\left\{\mu\, m\right\}\right) \mu^{-1}
\end{aligned} \tag{36}$$

F. We want to find the solution to the following integral...

$$\begin{aligned}
I_1 &= L_0 \int_m^n \text{Exp}\left\{-\lambda\, v\right\} \delta v \\
&= -\frac{1}{\lambda}\, L_0 \left(\text{Exp}\left\{-\lambda\, n\right\} - \text{Exp}\left\{-\lambda\, m\right\}\right) \\
&= L_0 \left(\text{Exp}\left\{-\lambda\, m\right\} - \text{Exp}\left\{-\lambda\, n\right\}\right) \lambda^{-1}
\end{aligned} \tag{37}$$

G. We want to find the solution to the following integral...

$$
\begin{aligned}
I &= \Delta^{-1} L_0^0 \left(\mu + \lambda\right)^{-1} \int_m^n \left(\text{Exp}\left\{\mu\, v\right\} - \text{Exp}\left\{-\lambda\, v\right\}\right) \delta v \\
&= \Delta^{-1} L_0^0 \left(\mu + \lambda\right)^{-1} \left[\frac{1}{\mu}\left(\text{Exp}\left\{\mu\, n\right\} - \text{Exp}\left\{\mu\, m\right\}\right) + \frac{1}{\lambda}\left(\text{Exp}\left\{-\lambda\, n\right\}\right) - \text{Exp}\left\{-\lambda\, m\right\}\right)\right] \\
&= \Delta^{-1} L_0^0 \left(\mu + \lambda\right)^{-1} \left[\left(\text{Exp}\left\{\mu\, n\right\} - \text{Exp}\left\{\mu\, m\right\}\right)\mu^{-1} + \left(\text{Exp}\left\{-\lambda\, n\right\}\right) - \text{Exp}\left\{-\lambda\, m\right\}\right)\lambda^{-1}\right]
\end{aligned} \tag{38}
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References

[1] Gary Schurman, *Integration By Parts - Weighted-Average Revenue Life,* January, 2020.